JOURNAL OF APPROXIMATION THEORY 72, 210–233 (1993)

Best Copositive Approximation

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Communicated by Günther Nürnberger

Received April 26, 1991; accepted in revised form December 3, 1991

A theory of best copositive approximation including a "zero in the convex hull" characterization, alternation theorem, and strong uniqueness theorem is developed for a general n-dimensional Chebyshev subspace in $C[a, b]$. In addition, a computational algorithm is discussed. © 1993 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to develop a theory for best uniform copositive approximation of continuous functions. Specifically, given a function $f \in C[a, b]$ and an n-dimensional Chebyshev subspace M of $C[a, b]$, we study

$$\inf\{\|f-q\| : q \in M \text{ and } f(x)\, q(x) \geqslant 0,\ x \in [a, b]\}.$$

The concept of best copositive approximation was introduced by Passow and Raymon in 1974 [4], where they showed that the best copositive approximation is unique if M is a subspace of all algebraic polynomials of degree $\leqslant n-1$. Moreover, they obtained a Jackson-type theorem for the degree of copositive approximation to f when f is a proper piecewise monotone function. Some other people [3, 6, 14] have also investigated this aspect. In the other direction, in 1975 Passow and Taylor [5] developed an alternation theory in the case when f does not change sign on any interval and M is an n-dimensional extended Chebyshev subspace of $C[a, b]$ of order 3. As a consequence, they obtained a "zero in the convex hull" characterization when the derivative of the best copositive approximation to f is nonzero at the points that f changes sign. This second result was extended by Shi [9] to the case that the sign changes of f are allowed on intervals but still the derivative of the best copositive approximation to f is nonzero at the points that f changes sign. In 1986

0021-9045/93 $5.00

Zhong [15] obtained a "zero in the convex hull" characterization theorem and an alternation theory under only the condition that M is an n-dimensional extended Chebyshev subspace of $C[a, b]$ of order 2. Also Zhong showed that the best copositive approximation is strongly unique.

However, for the case where M is a general n-dimensional Chebyshev subspace of $C[a, b]$, neither a "zero in the convex hull" characterization nor an alternation theory has been developed for the best copositive approximation. Besides, the uniqueness of best copositive approximation for this general case has not been known. In this paper, we give affirmative answers for the above problems. In Section 2 we give the definitions, notations, and basic facts that shall be used throughout this paper. In Section 3 three characterizations of best copositive approximations are established. A strong uniqueness theorem and the continuity of the best copositive approximation operator on a subset of $C[a, b]$ are developed in Section 4 and a computational algorithm is discussed in Section 5.

2. Definitions, Notations, and Basic Facts

Let M be an n-dimensional Chebyshev subspace of $C[a, b]$, i.e., every nonzero element of M has at most $n-1$ zeros. Let $f \in C[a, b]$ and define $M_f = \{p \in M : p(x) f(x) \geqslant 0 \text{ for all } x \in [a, b]\}$. If $p \in M_f$ has the property that

$$\|f - p\| = \inf\{\|f - q\| : q \in M_f\},$$

where $\|h\| = \max\{|h(x)| : x \in [a, b]\}$, then we say that p is a *best copositive approximation* (*from* M_f) *to* f.

Let $L(f) = \overline{\{x \in [a, b] : f(x) < 0\}}$, $U(f) = \overline{\{x \in [a, b] : f(x) > 0\}}$, and $S(f) = L(f) \cap U(f)$, where the bar denotes point set closure in the reals. We say that f changes sign at $t \in (a, b)$ if and only if $t \in S(f)$. On the other hand, we say that f changes sign on the interval $[c, d] \subset (a, b)$ if and only if $t \in [c, d]$ implies that $f(t) = 0$ with $c \in U(f)$ and $d \in L(f)$ (or $c \in L(f)$ and $d \in U(f)$).

For $f \in C[a, b]$, let the sets $SZ(f)$ and $DZ(f)$ of "simple" and "double" zeros be defined by $Z(f) = \{x \in [a, b] : f(x) = 0\}$, $DZ(f) = \{x \in Z(f)$: there exist $u, v \in [a, b]$ with $u < x < v$ such that f has constant nonzero sign on $[u, v] \backslash \{x\}\}$, and $SZ(f) = Z(f) \backslash DZ(f)$.

Denote the cardinality of a set A by $\operatorname{card}(A)$.

Theorem 2.1 (See [16, p. 24]). *Let M be an n-dimensional subspace of $C[a, b]$. Then M is Chebyshev if and only if* $\operatorname{card}(SZ(q)) + 2 \operatorname{card}(DZ(q)) \leqslant n - 1$ *for all* $q \in M \backslash \{0\}$.

THEOREM 2.2 (See [16, p. 25]). *Let M be an n-dimensional Chebyshev subspace of $C[a, b]$, and $s_1, ..., s_k, d_1, ..., d_l \in [a, b]$ distinct points with $s_1 < s_2 < \cdots < s_k$, $a < d_1 < \cdots < d_l < b$ and $0 \leqslant k + 2l \leqslant n - 1$. Then there is a $q \in M$ with $SZ(q) = \{s_1, ..., s_k\}$ and $DZ(q) = \{d_1, ..., d_l\}$ if at least one of the following conditions holds:*

(a) $n - (k + 2l)$ *is odd;*

(b) $a < s_1, s_k < b$;

(c) $a = s_1, s_k = b$.

COROLLARY 2.3. *If M is an n-dimensional Chebyshev subspace of $C[a, b]$ and $z_1, ..., z_m$ are m $(\leqslant n - 1)$ distinct points in (a, b), then there is a $q \in M$ such that q has simple zeros at these points and no other zeros.*

If the number of elements in $S(f)$ plus the number of times f changes sign on intervals is greater than or equal to n, then for any $q \in M_f$, q has at least n zeros, so $q(x) = 0$, therefore M_f consists of just the zero function.

If the number of elements in $S(f)$ plus the number of times f changes sign on intervals is less than n, let $z_1, ..., z_m$ be all the points at which f changes sign and $[z_{m+1}, u_{m+1}], ..., [z_N, u_N]$ denote all the intervals on which f changes sign. Then by Corollary 2.3, there is a $q \in M$ such that q has simple zeros at $z_1, ..., z_m, z_{m+1}, ..., z_N$ and no other zeros. Thus q or $-q \in M_f$. Hence M_f contains nontrivial functions.

Thus, throughout this paper, we always assume that the number of elements in $S(f)$ plus the number of times f changes sign on intervals is less than n. Let $S(f) = \{z_1, ..., z_m\}$, where $0 \leqslant m < n$.

To conclude this section, we state two well-known theorems.

THEOREM 2.4 (Theorem on Linear Inequalities [1, p. 19]). *Let U be a compact subset of R^n. Then there exists a $z \in R^n$ so that $(u, z) > 0$ for all $u \in U$ if and only if the origin of R^n does not belong to the convex hull of U.*

THEOREM 2.5 (Theorem of Carathéodory [1, p. 17]). *Let A be a subset of an n-dimensional linear space. Every point of the convex hull of A is expressible as a convex linear combination of $n + 1$ (or fewer) elements.*

3. CHARACTERIZATION OF BEST COPOSITIVE APPROXIMATIONS

LEMMA 3.1. *Let M be an n-dimensional Chebyshev subspace of $C[a, b]$. Then for any $p, q \in M \setminus \{0\}$ and $x \in [a, b]$,*

$$\lim_{\delta \to 0^+} \frac{p(x+\delta)}{q(x+\delta)} \quad \textit{and} \quad \lim_{\delta \to 0^+} \frac{p(x-\delta)}{q(x-\delta)}$$

always exist as extended real numbers.

Proof. Without loss of generality, we may assume that $q(x+\delta)>0$ for $0<\delta\leqslant\delta_0$.

Suppose we have

$$\liminf_{\delta\to 0^+}\frac{p(x+\delta)}{q(x+\delta)}=\alpha, \qquad \limsup_{\delta\to 0^+}\frac{p(x+\delta)}{q(x+\delta)}=\beta,$$

and $\alpha<\beta$.

Let $\alpha<\gamma<\beta$. Then there exist strictly decreasing sequences $\delta_k\to 0^+$ and $\varepsilon_k\to 0^+$ such that

$$\lim_{k\to\infty}\frac{p(x+\delta_k)}{q(x+\delta_k)}=\alpha<\gamma<\beta=\lim_{k\to\infty}\frac{p(x+\varepsilon_k)}{q(x+\varepsilon_k)}.$$

Hence, there exists $N>0$ such that

$$\frac{p(x+\delta_k)}{q(x+\delta_k)}<\gamma \qquad\text{and}\qquad \frac{p(x+\varepsilon_k)}{q(x+\varepsilon_k)}>\gamma \qquad\text{for all}\quad k\geqslant N.$$

Thus, $p(x+\delta_k)-\gamma q(x+\delta_k)<0$ and $p(x+\varepsilon_k)-\gamma q(x+\varepsilon_k)>0$ for all $k\geqslant N$. So $p-\gamma q$ has more than n zeros and thus $p=\gamma q$, which contradicts $\alpha<\beta$. Hence, $\alpha=\beta$. That is,

$$\lim_{\delta\to 0^+}\frac{p(x+\delta)}{q(x+\delta)}\text{ exists.}$$

Similar, we have

$$\lim_{\delta\to 0^+}\frac{p(x-\delta)}{q(x-\delta)}\text{ exists.} \quad \blacksquare$$

LEMMA 3.2. *Let* $f\in C[a,b]$ *and* $M_0=\{q\in M : q(x)=0, x\in S(f)\}$. *If* $z_1, ..., z_m$ *are all the points at which* f *changes sign and* $[z_{m+1}, u_{m+1}], ..., [z_N, u_N]$ $(N\leqslant n-1)$ *are all the intervals on which* f *changes sign, then there exist* $\phi_1, ..., \phi_{n-m}\in M_0$ *and* $b_0\in(\max\{z_1, ..., z_N, u_{m+1}, ..., u_N\}, b]$ *such that* $\phi_i\in M_f|_{[a,b_0]}$ *(i.e.* $\phi_i(x)f(x)\geqslant 0$ *for all* $x\in[a,b_0]$, $i=1, ..., n-m$*) and* $M_0=\operatorname{span}\{\phi_1, ..., \phi_{n-m}\}$.

Proof. Since $S(f)=\{z_1, ..., z_m\}$ consists of m distinct points, the dimension of M_0 is $n-m$. Choose $\{z_i\}_{i=N+1}^{n-1}$ and $\{u_i\}_{i=N+1}^{n-1}$ in $(\max\{z_1, ..., z_N, u_{m+1}, ..., u_N\}, b]$ so that

$$z_{N+1}<u_{N+1}<z_{N+2}<u_{N+2}<\cdots<z_{n-1}<u_{n-1}.$$

Select $\phi_i\in M\setminus\{0\}$, $i=m+1, ..., n-1$, such that

$$\phi_i(u_i)=\phi_i(z_j)=0 \qquad\text{for}\quad j=1, ..., i-1, i+1, ..., n-1.$$

Hence ϕ_i changes sign at $z_1, ..., z_m, z_{m+1}, ..., z_{i-1}, u_i, z_{i+1}, ..., z_{n-1}$. Also, we choose $\phi_n \in M\setminus\{0\}$ such that $\phi_n(z_i) = 0$, $i = 1, ..., n-1$. It is clear that we may assume

$$\phi_i(x) f(x) \geqslant 0, \qquad \text{for all} \quad x \in [a, z_{N+1}), i = m+1, ..., n.$$

Next we shall show that $\phi_{m+1}, ..., \phi_n$ are linearly independent. Assume that

$$c_{m+1}\phi_{m+1} + \cdots + c_n\phi_n = 0.$$

Since $\phi_i(z_i) \neq 0$ for $i = m+1, ..., n-1$, and

$$c_{m+1}\phi_{m+1}(z_i) + \cdots + c_n\phi_n(z_i) = c_i\phi_i(z_i) = 0,$$

$c_i = 0$, $i = m+1, ..., n-1$. So $c_n\phi_n = 0$ and thus $c_n = 0$. Therefore $\phi_{m+1}, ..., \phi_n$ are linearly independent. Let $b_0 = z_{N+1}$. Then $\phi_i \in M_f|_{[a, b_0]}$, $i = m+1, ..., n$, and $M_0 = \text{span}\{\phi_{m+1}, ..., \phi_n\}$. This completes the proof. ∎

LEMMA 3.3. *Assume that* $M_0 = \text{span}\{\phi_1, ..., \phi_{n-m}\}$, *where* $\phi_1, ..., \phi_{n-m} \in M_f|_{[a, b_0]}$. *Let* $\Phi = \sum_{i=1}^{n-m} \phi_i$. *Then for any* $h \in M_0$, *there is a* $\lambda_0 > 0$ *such that* $\Phi + \lambda h \in M_f|_{[a, b_0]}$ *for* $0 < \lambda \leqslant \lambda_0$.

Proof. Since $M_0 = \text{span}\{\phi_1, ..., \phi_{n-m}\}$ for any $h \in M_0$, we may write $h = \sum_{i=1}^{n-m} a_i\phi_i$. Hence,

$$\Phi + \lambda h = \sum_{i=1}^{n-m} (1 + \lambda a_i)\, \phi_i.$$

Choose $\lambda_0 > 0$ small enough so that $1 + \lambda a_i > 0$, $i = 1, ..., n-m$. Then $\Phi + \lambda h \in M_f|_{[a, b_0]}$ for $0 < \lambda \leqslant \lambda_0$. ∎

LEMMA 3.4. *If, for some* $x \in S(f)$ *and* $p \in M_f$,

$$\lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} > 0 \qquad \left(\text{or } \lim_{\delta \to 0^+} \frac{p(x-\delta)}{\Phi(x-\delta)} > 0\right),$$

then for any $h \in M_0$, *there exist* $\lambda_0 > 0$ *and* $\delta_0 > 0$ *such that*

$$f(x+\delta)[p(x+\delta) + \lambda h(x+\delta)] \geqslant 0 \qquad (\text{or } f(x-\delta)[p(x-\delta) + \lambda h(x-\delta)] \geqslant 0)$$

for all $0 < \lambda \leqslant \lambda_0$ *and* $0 < \delta < \delta_0$.

Proof. Without loss of generality, we may assume that f is increasing at $x \in S(f)$. Since

$$\lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} > 0,$$

there exists $\lambda_1 > 0$ such that

$$\lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} > \lambda_1 > 0.$$

So there exists $\delta_0 > 0$ such that

$$\frac{p(x+\delta)}{\Phi(x+\delta)} > \lambda_1 > 0 \qquad \text{for} \quad 0 < \delta < \delta_0.$$

By Lemma 3.3, for any $h \in M_0$, there exists $\lambda_2 > 0$ such that

$$\Phi(x+\delta) + \lambda_2 h(x+\delta) > 0 \qquad \text{for all} \quad 0 < \delta \leqslant \delta_0$$

so

$$p(x+\delta) + \lambda_1 \lambda_2 h(x+\delta) > 0 \qquad \text{for all} \quad 0 < \delta \leqslant \delta_0.$$

Letting $\lambda_0 = \lambda_1 \lambda_2$, we have

$$f(x+\delta)[p(x+\delta) + \lambda h(x+\delta)] \geqslant 0$$

for all $0 < \lambda \leqslant \lambda_0$, $0 < \delta < \delta_0$. ∎

LEMMA 3.5. *Let* $M_0 = \text{span}\{\phi_1, ..., \phi_{n-m}\}$, *where* $\phi_1, ..., \phi_{n-m} \in M_f|_{[a,b_0]}$, *and* $\Phi(x) = \sum_{i=1}^{n-m} \phi_i(x)$. *Then*

$$\lim_{\delta \to 0^+} \frac{\phi_i(x+\delta)}{\Phi(x+\delta)} \leqslant 1 \qquad \textit{and} \qquad \lim_{\delta \to 0^+} \frac{\phi_i(x-\delta)}{\Phi(x-\delta)} \leqslant 1 \qquad \textit{for any} \quad x \in S(f).$$

Proof. Without loss of generality, we may assume that $\phi_i(x+\delta) > 0$ for $0 < \delta \leqslant \delta_0$, $i = 1, ..., n-m$. Then we have

$$\Phi(x+\delta) = \sum_{i=1}^{n-m} \phi_i(x+\delta) \geqslant \phi_i(x+\delta) \qquad \text{if} \quad 0 < \delta \leqslant \delta_0,$$

so

$$\frac{\phi_i(x+\delta)}{\Phi(x+\delta)} \leqslant 1, \qquad 0 < \delta \leqslant \delta_0, \qquad \text{and thus} \qquad \lim_{\delta \to 0^+} \frac{\phi_i(x+\delta)}{\Phi(x+\delta)} \leqslant 1.$$

Similarly, we have

$$\lim_{\delta \to 0^+} \frac{\phi_i(x-\delta)}{\Phi(x-\delta)} \leqslant 1. \quad ∎$$

Throughout this paper, we always assume that $M_0 = \text{span}\{\phi_1, ..., \phi_{n-m}\}$, where $\phi_1, ..., \phi_{n-m} \in M_f|_{[a,b_0]}$ and b_0 is as in Lemma 3.2. Let $\Phi(x) = \sum_{i=1}^{n-m} \phi_i(x)$.

For $p \in M_f$, define

$$X_{+1} = X_{+1}(p) = \{x \in [a, b] : f(x) - p(x) = \|f - p\|\}$$
$$X_{-1} = X_{-1}(p) = \{x \in [a, b] : f(x) - p(x) = -\|f - p\|\}$$
$$X_{+2} = X_{+2}(p) = \{x \in U(f) \backslash S(f) : p(x) = 0, |f(x)| < \|f - p\|\}$$
$$X_{-2} = X_{-2}(p) = \{x \in L(f) \backslash S(f) : p(x) = 0, |f(x)| < \|f - p\|\}$$
$$X_3 = X_3(p) = \left\{x \in S(f) : \lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} = 0\right\}$$
$$X_4 = X_4(p) = \left\{x \in S(f) : \lim_{\delta \to 0^+} \frac{p(x-\delta)}{\Phi(x-\delta)} = 0\right\}$$
$$X_p^* = X_{+1} \cup X_{-1} \cup X_{+2} \cup X_{-2}$$
$$X_p^{**} = X_3 \cup X_4$$
$$X_p = X_p^* \cup X_p^{**}$$
$$\sigma(x) = \begin{cases} 1 & \text{if } \ x \in X_{+1} \cup X_{+2} \\ -1 & \text{if } \ x \in X_{-1} \cup X_{-2} \\ \operatorname{sgn} \Phi(x+\delta_0) & \text{if } \ x \in X_3 \cup X_4, \text{ where } \Phi(x+\delta) \neq 0 \\ & \quad \text{for all } 0 < \delta \leqslant \delta_0. \end{cases}$$

LEMMA 3.6 (See [15]). *Let $f \in C[a, b] \backslash M$. Then the best copositive approximation to f is nonzero.*

Proof. Assume that $p(x) \equiv 0$ is a best copositive approximation to f. By the Kolmogorov criterion, we have

$$\max_{x \in X_{+1} \cup X_{-1}} (-q(x)) f(x) \geqslant 0 \qquad \text{for all} \quad q \in M_f.$$

Let $z_1, ..., z_m$ be all the points at which f changes sign and $[z_{m+1}, u_{m+1}], ..., [z_N, u_N]$ $(N \leqslant n-1)$ denote all the intervals on which f changes sign. Let $t_i \in (z_i, u_i)$, $i = m+1, ..., N$. Choose $q_0 \in M$ such that q_0 has simple zeros at $z_1, ..., z_m, t_{m+1}, ..., t_N$ and no other zeros. Set $q_1 = \sigma q_0$ and select $\sigma = +1$ or -1 appropriately so that $q_1 \in M_f$. It is obvious that for any $x \in X_{+1} \cup X_{-1}$, $x \notin \{z_1, ..., z_m, t_{m+1}, ..., t_N\}$. Since $|f(x)| = \|f\| > 0$ for all $x \in X_{+1} \cup X_{-1}$, $q_1(x) f(x) > 0$ for all $x \in X_{+1} \cup X_{-1}$. Thus $\max_{x \in X_{+1} \cup X_{-1}} (-q_1(x)) f(x) < 0$. This is a contradiction. ∎

LEMMA 3.7. *Let $p \in M_f \backslash \{0\}$. Suppose $q \in M_0$ satisfies*

$$\lim_{\delta \to 0^+} \frac{q(x+\delta)}{\Phi(x+\delta)} > 0 \quad \textit{for} \quad x \in X_3, \qquad \lim_{\delta \to 0^+} \frac{q(x-\delta)}{\Phi(x-\delta)} > 0 \quad \textit{for} \quad x \in X_4,$$

and

$$\operatorname{sgn} q(x) = sgf(x) \qquad \textit{if} \quad x \in A_0 = \{x \in L(f) \cup U(f) \backslash S(f) : p(x) = 0\},$$

where

$$sgf(x) = \begin{cases} 1 & x \in U(f) \backslash S(f), \\ -1 & x \in L(f) \backslash S(f), \\ 0 & x \notin L(f) \cup U(f) \backslash S(f). \end{cases}$$

Then there exists $\lambda_0 > 0$ *such that* $p + \lambda q \in M_f$ $(0 < \lambda \leqslant \lambda_0)$.

Proof. (1) If $x \in A_0$, then $\operatorname{sgn} q(x) = sgf(x)$ so there exists a neighborhood Δ_x of x such that

$$f(t)\, q(t) \geqslant 0 \qquad \text{for all} \quad t \in \Delta_x.$$

(2) For $x \in X_3 \cup X_4$, we consider two cases.

(a) If $x \in X_3 \backslash X_4$, then

$$\lim_{\delta \to 0^+} \frac{q(x+\delta)}{\Phi(x+\delta)} > 0, \qquad \lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} = 0, \qquad \text{and} \qquad \lim_{\delta \to 0^+} \frac{p(x-\delta)}{\Phi(x-\delta)} > 0.$$

Since

$$\lim_{\delta \to 0^+} \frac{q(x+\delta)}{\Phi(x+\delta)} > 0,$$

there exists $\delta_0 > 0$ such that

$$f(x+\delta)\, q(x+\delta) \geqslant 0 \qquad \text{for all} \quad 0 < \delta \leqslant \delta_0.$$

And since

$$\lim_{\delta \to 0^+} \frac{p(x-\delta)}{\Phi(x-\delta)} > 0,$$

Lemma 3.4 implies that there exist $\lambda_0 > 0$ and $\delta_0 > 0$ such that

$$f(x-\delta)[p(x-\delta) + \lambda q(x-\delta)] \geqslant 0 \qquad \text{for all} \quad 0 < \delta \leqslant \delta_0,\ 0 < \lambda \leqslant \lambda_0.$$

Hence, for $x \in X_3 \backslash X_4$, there exist a neighborhood Δ_x of x and a number $\lambda_0 > 0$ such that

$$f(t)[p(t) + \lambda q(t)] \geqslant 0, \qquad t \in \Delta_x,\ 0 < \lambda \leqslant \lambda_0.$$

Similarly, if $x \in X_4 \backslash X_3$, there exist a neighborhood Δ_x of x and a number $\lambda_0 > 0$ such that

$$f(t)[p(t) + \lambda q(t)] \geqslant 0, \qquad t \in \Delta_x, 0 < \lambda \leqslant \lambda_0.$$

(b) If $x \in X_3 \cap X_4$, then

$$\lim_{\delta \to 0^+} \frac{q(x+\delta)}{\Phi(x+b)} > 0 \qquad \text{and} \qquad \lim_{\delta \to 0^+} \frac{q(x-\delta)}{\Phi(x-\delta)} > 0.$$

Hence, there exists a neighborhood Δ_x of x such that

$$f(t)\, q(t) \geqslant 0 \qquad \text{for all} \quad t \in \Delta_x.$$

(3) If $x \in S(f) \backslash (X_3 \cup X_4)$, then

$$\lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} > 0 \qquad \text{and} \qquad \lim_{\delta \to 0^+} \frac{p(x-\delta)}{\Phi(x-\delta)} > 0.$$

By Lemma 3.4, there exist a neighborhood Δ_x of x and a number $\lambda_0 > 0$ such that

$$f(t)[p(t) + \lambda q(t)] \geqslant 0, \qquad t \in \Delta_x, 0 < \lambda \leqslant \lambda_0.$$

Combining (1), (2), and (3), we have that, for any $x \in A_0 \cup S(f)$, there exist a neighborhood Δ_x of x and a number $\lambda_0 > 0$ such that

$$f(t)[p(t) + \lambda q(t)] \geqslant 0, \qquad t \in \Delta_x, 0 < \lambda \leqslant \lambda_0.$$

Let $\{[a_i, b_i]\}_{i \in I}$ be the set of all the intervals on which f vanishes. And let $\Delta_i = (a_i, b_i)$, $I \in I$. Then for any $t \in \bigcup_{i \in I} \Delta_i$, $f(t) = 0$. So

$$f(t)[p(t) + \lambda q(t)] \geqslant 0 \qquad \text{for} \quad t \in \bigcup_{i \in I} \Delta_i, 0 < \lambda \leqslant \lambda_0.$$

For $t \in H = [a, b] \backslash [(\bigcup_{x \in A_0 \cup S(f)} \Delta_x) \cup (\bigcup_{i \in I} \Delta_i)]$, we have $p(t) \neq 0$.
Since H is a compact set, there exists $\lambda_0 > 0$ such that

$$\lambda_0\, |q(t)| < |p(t)| \qquad \text{for all} \quad t \in H.$$

Thus

$$f(t)[p(t) + \lambda q(t)] \geqslant 0 \qquad \text{for} \quad t \in H, 0 < \lambda \leqslant \lambda_0.$$

But

$$f(t)[p(t) + \lambda q(t)] \geqslant 0 \qquad \text{for} \quad t \in \left(\bigcup_{x \in A_0 \cup S(f)} \Delta_x \right) \cup \left(\bigcup_{i \in I} \Delta_i \right), 0 < \lambda \leqslant \lambda_0.$$

Hence,

$$p+\lambda q \in M_f, \qquad 0<\lambda \leqslant \lambda_0. \quad \blacksquare$$

THEOREM 3.8. (The "Zero in the Convex Hull" Characterization). *Let M be an n-dimensional Chebyshev subspace of $C[a,b]$. Suppose that $f \in C[a,b] \backslash M$, $M_0 = \operatorname{span}\{\phi_1, ..., \phi_{n-m}\}$, where $\phi_1, ..., \phi_{n-m} \in M_f|_{[a,b_0]}$. Let $\Phi(x) = \sum_{i=1}^{n-m} \phi_i(x)$ and*

$$\begin{aligned} B = &\{\sigma(x)(\phi_1(x), ..., \phi_{n-m}(x)) : x \in X_p^*\} \\ &\cup \left\{\left(\lim_{\delta \to 0^+} \frac{\phi_1(x+\delta)}{\Phi(x+\delta)}, ..., \lim_{\delta \to 0^+} \frac{\phi_{n-m}(x+\delta)}{\Phi(x+\delta)}\right) : x \in X_3\right\} \\ &\cup \left\{\left(\lim_{\delta \to 0^+} \frac{\phi_1(x-\delta)}{\Phi(x-\delta)}, ..., \lim_{\delta \to 0^+} \frac{\phi_{n-m}(x-\delta)}{\Phi(x-\delta)}\right) : x \in X_4\right\}. \end{aligned}$$

Then p is a best copositive approximation to f if and only if $p \not\equiv 0$ and the origin of the $(n-m)$-dimensional Euclidean space belongs to the convex hull of B.

Proof. ($\Leftarrow$) Suppose that p is not a best copositive approximation to f. Then there exists $p_1 \in M_f$ such that $\|f-p_1\| < \|f-p\|$.

Let $q_1 \in M_f$ be chosen as in the proof of Lemma 3.6. Then

$$\sigma(x)\, q_1(x) > 0 \qquad \text{for all} \quad x \in X_{+2} \cup X_{-2}.$$

Choose $\lambda_0 > 0$ so that

$$\lambda_0\, |\Phi(x)| < |q_1(x)| \qquad \text{for all} \quad x \in X_{+2} \cup X_{-2}.$$

Let $q(x) = q_1(x) + \lambda_0 \Phi(x)$. Choose $\lambda > 0$ small enough such that

$$\|f - p_1 - \lambda q\| < \|f - p\|.$$

Then we have that

(1) If $x \in X_{+1} \cup X_{-1}$, then

$$\begin{aligned} &|f(x) - p_1(x) - \lambda q(x)| < |f(x) - p(x)| \\ &\quad \Rightarrow [f(x) - p(x)][(f(x) - p(x)) - (f(x) - p_1(x) - \lambda q(x))] > 0 \\ &\quad \Rightarrow [f(x) - p(x)][p_1(x) + \lambda q(x) - p(x)] > 0. \end{aligned}$$

Let $h(x) = p_1(x) + \lambda q(x) - p(x)$. Then $h \in M_0$ and

$$\sigma(x)\, h(x) > 0, \qquad \text{for} \quad x \in X_{+1} \cup X_{-1}.$$

(2) If $x \in X_{+2} \cup X_{-2}$, then

$$\sigma(x)\, h(x) = \sigma(x)\, p_1(x) + \lambda\sigma(x)\, q(x) \geqslant \lambda\sigma(x)\, q(x) > 0.$$

Thus

$$\sigma(x)\, h(x) > 0 \qquad \text{for} \quad x \in X_{+2} \cup X_{-2}.$$

(3) If $x \in X_3$, then

$$\lim_{\delta \to 0^+} \frac{p(x+\delta)}{\Phi(x+\delta)} = 0.$$

Hence,

$$\begin{aligned}
\lim_{\delta \to 0^+} \frac{h(x+\delta)}{\Phi(x+\delta)} &= \lim_{\delta \to 0^+} \frac{p_1(x+\delta) + \lambda q(x+\delta) - p(x+\delta)}{\Phi(x+\delta)} \\
&= \lim_{\delta \to 0^+} \left[\frac{p_1(x+\delta)}{\Phi(x+\delta)} + \lambda \left(\frac{q_1(x+\delta)}{\Phi(x+\delta)} + \lambda_0 \right) - \frac{p(x+\delta)}{\Phi(x+\delta)} \right] \\
&= \lim_{\delta \to 0^+} \frac{p_1(x+\delta)}{\Phi(x+\delta)} + \lambda \lim_{\delta \to 0^+} \frac{q_1(x+\delta)}{\Phi(x+\delta)} + \lambda\lambda_0 \geqslant \lambda\lambda_0 > 0.
\end{aligned}$$

Therefore,

$$\lim_{\delta \to 0^+} \frac{h(x+\delta)}{\Phi(x+\delta)} > 0 \qquad \text{for all} \quad x \in X_3.$$

(4) If $x \in X_4$, using a similar argument as in (3), we have

$$\lim_{\delta \to 0^+} \frac{h(x-\delta)}{\Phi(x-\delta)} > 0 \qquad \text{for all} \quad x \in X_4.$$

Combining (1), (2), (3), and (4), we have that

$$\begin{aligned}
\sigma(x)\, h(x) &> 0 \qquad \text{if} \quad x \in X_p^*, \\
\lim_{\delta \to 0^+} \frac{h(x+\delta)}{\Phi(x+\delta)} &> 0 \qquad \text{if} \quad x \in X_3, \\
\lim_{\delta \to 0^+} \frac{h(x-\delta)}{\Phi(x-\delta)} &> 0 \qquad \text{if} \quad x \in X_4.
\end{aligned}$$

Since $X_p^* \cup X_3 \cup X_4$ is compact, by the Theorem on Linear Inequalities, the origin of R^{n-m} does not belong to the convex hull of B. This is a contradiction.

($\Rightarrow$) Let p be a best copositive approximation to f. By Lemma 3.6, $p(x) \not\equiv 0$. In the remainder of our proof, we shall show that the origin of R^{n-m} belongs to the convex hull of B.

Suppose not, by the Theorem on Linear Inequalities, there is a $q \in M_0$ such that

$$\sigma(x)\, q(x) > 0 \qquad \text{if} \quad x \in X_p^*,$$

$$\lim_{\delta \to 0^+} \frac{q(x+\delta)}{\Phi(x+\delta)} > 0 \qquad \text{if} \quad x \in X_3,$$

$$\lim_{\delta \to 0^+} \frac{q(x-\delta)}{\Phi(x-\delta)} > 0 \qquad \text{if} \quad x \in X_4.$$

Let $q_\lambda(x) = p(x) + \lambda q(x)$. By Lemma 3.7, there exists $\lambda_0 > 0$ such that $q_\lambda \in M_f$ for $0 < \lambda \leqslant \lambda_0$.

On the other hand, by the continuity of the functions, there exist an open set $G \supset X_{+1} \cup X_{-1}$ and $\lambda_1 > 0$ such that

$$\lambda_1\, |q(x)| < |f(x) - p(x)| \quad \text{and} \quad q(x) \operatorname{sgn}[f(x) - p(x)] > 0 \quad \text{for } x \in G.$$

Hence, for $x \in G$ and $0 < \lambda \leqslant \lambda_1$, we have

$$\begin{aligned} |f(x) - q_\lambda(x)| &= |f(x) - p(x) - \lambda q(x)| \\ &= \operatorname{sgn}[f(x) - p(x)][f(x) - p(x) - \lambda q(x)] \\ &= |f(x) - p(x)| - \lambda q(x) \operatorname{sgn}[f(x) - p(x)] \\ &< |f(x) - p(x)| \leqslant \|f - p\|. \end{aligned}$$

For $x \in [a, b] \backslash G$, $|f(x) - p(x)| < \|f - p\|$.

Choose $\lambda_2 > 0$ small enough so that

$$|f(x) - q_\lambda(x)| < \|f - p\| \qquad \text{for all} \quad x \in [a, b] \backslash G,\ 0 < \lambda \leqslant \lambda_2.$$

Let $\lambda_3 = \min\{\lambda_0, \lambda_1, \lambda_2\}$. Then

$$q_{\lambda_3} \in M_f \qquad \text{and} \qquad \|f - q_{\lambda_3}\| < \|f - p\|.$$

This contradicts the fact that p is a best copositive approximation to f. ∎

DEFINITION 3.9. Let $x, y \in X_p$, $x < y$, and $(x, y] \cap S(f) = \{z_1, ..., z_v\}$, $v \geqslant 0$, where $v = 0$ implies $(x, y] \cap S(f) = \varnothing$. We say that $f - p$ alternates once in (x, y) if $\sigma(x) = (-1)^{v+1} \sigma(y)$.

THEOREM 3.10. *Let $f \in C[a, b] \backslash M$. Then $p \in M_f$ is a best copositive approximation to f if and only if $p(x) \not\equiv 0$ and there exist $n - m + 1$*

constants $\lambda_v > 0$, $v = 1, ..., n-m+1$, *and* $n-m+1$ *distinct points* $\{x_v\}_{v=1}^{N_0} \subset X_{+1} \cup X_{-1}$, $\{x_v\}_{v=N_0+1}^{N_1} \subset X_{+2} \cup X_{-2}$, $\{x_v\}_{v=N_1+1}^{N_2} \subset X_3$, *and* $\{x_v\}_{v=N_2+1}^{n-m+1} \subset X_4$, *where* $N_0 \geq 1$, *such that*

$$\sum_{v=1}^{N_1} \lambda_v \sigma(x_v)\, q(x_v) + \sum_{v=N_1+1}^{N_2} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v+\delta)}{\Phi(x_v+\delta)} + \sum_{v=N_2+1}^{n-m+1} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v-\delta)}{\Phi(x_v-\delta)} = 0$$

for all $q \in M_0$.

Proof. ($\Leftarrow$) It is clear that the origin of R^{n-m} belongs to the convex hull of B. By Theorem 3.8, p is a best copositive approximation to f.

($\Rightarrow$) If p is a best copositive approximation of f, then it follows from Theorem 3.8 that $p(x) \not\equiv 0$ and the origin of R^{n-m} belongs to the convex hull of B. By the Theorem of Carathéodory, there exist k constants $\lambda_v > 0$, $v = 1, ..., k$, and k points $\{x_v\}_{v=1}^{N_0} \subset X_{+1} \cup X_{-1}$, $\{x_v\}_{v=N_0+1}^{N_1} \subset X_{+2} \cup X_{-2}$, $\{x_v\}_{v=N_1+1}^{N_2} \subset X_3$, and $\{x_v\}_{v=N_2+1}^{k} \subset X_4$, where $k \leq n-m+1$, such that

$$\sum_{v=1}^{N_1} \lambda_v \sigma(x_v)\, q(x_v) + \sum_{v=N_1+1}^{N_2} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v+\delta)}{\Phi(x_v+\delta)} + \sum_{v=N_2+1}^{k} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v-\delta)}{\Phi(x_v-\delta)} = 0 \tag{1}$$

for all $q \in M_0$.

Let $W = \{x_1, ..., x_k\} = \{w_1, ..., w_{k_0}\}$, where $w_1 < \cdots < w_{k_0}$, $k_0 \leq k$.

Next we shall show that $k_0 = n-m+1$.

Suppose that $k_0 \leq n-m$. If $f-p$ alternates once in (w_i, w_{i+1}) for some $1 \leq i \leq k_0 - 1$, we choose $s_i \in (w_i, w_{i+1}) \backslash S(f)$. Let $Z_0 = \{s_i \in (w_i, w_{i+1}) \backslash S(f) : f-p$ alternates once in (w_i, w_{i+1}), $1 \leq i \leq k_0 - 1\}$. If $Z = Z_0 \cup S(f)$, then Z consists of at most $k_0 - 1 + m \leq n-m-1+m = n-1$ elements.

Now choose $q \in M$ such that q has simple zeros at the points of Z and no other zeros. Choose $\lambda > 0$ so that $q(x) \neq 0$ for all $x \in \bigcup_{i=1}^{k_0} (w_i, w_i + \lambda] \cup \bigcup_{z \in S(f)} (z, z+\lambda]$. We may assume that $\sigma(w_1) = \operatorname{sgn} q(w_1 + \lambda)$ and $(w_1, w_2] \cap S(f) = \{z_{11}, ..., z_{1v}\}$. If $f-p$ alternates once in (w_1, w_2), then $\sigma(w_2) = (-1)^{v+1} \sigma(w_1)$ and q has simple zeros $z_{11}, ..., z_{1v}, s_1$ in $(w_1, w_2]$. Thus $\operatorname{sgn} q(w_2 + \lambda) = (-1)^{v+1} \operatorname{sgn} q(w_1 + \lambda)$ and hence

$$\sigma(w_2) = \operatorname{sgn} q(w_2 + \lambda).$$

If $f-p$ does not alternate once in (w_1, w_2), then $\sigma(w_2)=(-1)^v \sigma(w_1)$ and q has simple zeros $z_{11}, ..., z_{1v}$ in $(w_1, w_2]$. Thus $\operatorname{sgn} q(w_2+\lambda)=(-1)^v \operatorname{sgn} q(w_1+\lambda)$ and hence

$$\sigma(w_2)=\operatorname{sgn} q(w_2+\lambda).$$

Similarly, we have

$$\sigma(w_i)=\operatorname{sgn} q(w_i+\lambda), \qquad i=1, ..., k_0.$$

Therefore

$$\sigma(x_v)\, q(x_v)>0, \qquad v=1, ..., N_1,$$

$$\lim_{\delta \to 0^+} \frac{q(x_v+\delta)}{\Phi(x_v+\delta)} \geqslant 0, \qquad v=N_1+1, ..., N_2,$$

$$\lim_{\delta \to 0^+} \frac{q(x_v-\delta)}{\Phi(x_v-\delta)} \geqslant 0, \qquad v=N_2+1, ..., k.$$

Choose $\lambda_0>0$ so that

$$\lambda_0\, |\Phi(x_v)|<|q(x_v)|, \qquad v=1, ..., N_1.$$

Let $q_0=q+\lambda_0 \Phi$. Then $q_0 \in M_0$ and

$$\sigma(x_v)\, q_0(x_v)>0, \qquad v=1, ..., N_1,$$

$$\lim_{\delta \to 0^+} \frac{q_0(x_v+\delta)}{\Phi(x_v+\delta)}>0, \qquad v=N_1+1, ..., N_2,$$

$$\lim_{\delta \to 0^+} \frac{q_0(x_v-\delta)}{\Phi(x_v-\delta)}>0, \qquad v=N_2+1, ..., k.$$

Therefore q_0 does not satisfy Eq. (1). So we have $k=k_0=n-m+1$.

Let $q_1 \in M_f$ be chosen as in the proof of Lemma 3.6. Then

$$\sigma(x)\, q_1(x)>0 \qquad \text{for all} \quad x \in X_{+2} \cup X_{-2}.$$

Choose $\lambda_0>0$ so that

$$\lambda_0\, |\Phi(x)|<|q_1(x)| \qquad \text{for all} \quad x \in X_{+2} \cup X_{-2}.$$

Let $q_2=q_1+\lambda_0 \Phi$. Then $q_2 \in M_0$ and

$$\sigma(x)\, q_2(x)>0 \qquad \text{if} \quad x \in X_{+2} \cup X_{-2},$$

$$\lim_{\delta \to 0^+} \frac{q_2(x+\delta)}{\Phi(x+\delta)}>0 \qquad \text{if} \quad x \in X_3,$$

$$\lim_{\delta \to 0^+} \frac{q_2(x-\delta)}{\Phi(x-\delta)}>0 \qquad \text{if} \quad x \in X_4.$$

Therefore $N_0 \geqslant 1$. ∎

THEOREM 3.11 (The Alternation Theorem). *Let* $f \in C[a, b] \backslash M$ *and* $p \in M_f$. *Then* p *is a best copositive approximation to* f *if and only if* $p(x) \not\equiv 0$ *and there exist* $n-m+1$ *points* $w_1 < \cdots < w_{n-m+1}$ *in* X_p *such that* $f-p$ *alternates once in each* (w_i, w_{i+1}), $i = 1, ..., n-m$.

Proof. ($\Leftarrow$) Suppose that there exist $n-m+1$ points $w_1 < \cdots < w_{n-m+1}$ in X_p such that $f-p$ alternates once in each (w_i, w_{i+1}), $i = 1, ..., n-m$. If p were not a best copositive approximation to f, by Theorem 3.8 and the Theorem on Linear Inequalities, there would exist a $h \in M_0$ such that

$$\sigma(x)\, h(x) > 0 \qquad \text{if} \quad x \in {}^{*}_{p}, \tag{2}$$

$$\lim_{\delta \to 0^+} \frac{h(x+\delta)}{\Phi(x+\delta)} > 0 \qquad \text{if} \quad x \in X_3, \tag{3}$$

$$\lim_{\delta \to 0^+} \frac{h(x-\delta)}{\Phi(x-\delta)} > 0 \qquad \text{if} \quad x \in X_4. \tag{4}$$

Let

$$DZ(h) \cap X_3 = \{d_1, ..., d_k\} \qquad \text{and} \qquad DZ(h) \cap X_4 = \{d_{k+1}, ..., d_{k_1}\}.$$

It is clear that there is a $\delta_0 > 0$ such that

$$(d_i - \delta_0, d_i) \cap (X_p \cup Z(h)) = \varnothing, \qquad i = 1, ..., k,$$
$$(d_i, d_i + \delta_0) \cap (X_p \cup Z(h)) = \varnothing, \qquad i = k+1, ..., k_1.$$

Now choose $h_0 \in M$ such that

$$\begin{aligned} SZ(h_0) = SZ(h) &\cup \{d_1, ..., d_k, d_{k+1}, ..., d_{k_1}\} \\ &\cup \{d_1 - \delta_0, ..., d_k - \delta_0, d_{k+1} + \delta_0, ..., d_{k_1} + \delta_0\}, \\ DZ(h_0) = DZ(h) &\backslash \{d_1, ..., d_k, d_{k+1}, ..., d_{k_1}\}. \end{aligned}$$

Let $q = \sigma h_0$. Then we may appropriately select $\sigma = 1$ or $\sigma = -1$ such that

$$\operatorname{sgn} q(x) = \operatorname{sgn} h(x), \quad \text{for } x \in [a, b] \Big\backslash \bigcup_{i=1}^{k} (d_i - \delta_0, d_i) \cup \bigcup_{i=k+1}^{k_1} (d_i, d_i + \delta_0).$$

Hence, from (2), (3), (4), and the fact that q has simple zeros at the points in $X_3 \cup X_4$, we have

$$\sigma(x)\, q(x) > 0 \qquad \text{if} \quad x \in X_p^*, \tag{5}$$

$$\sigma(x)\, q(x+\delta) > 0 \qquad \text{if} \quad x \in X_3, 0 < \delta \leqslant \delta_0, \tag{6}$$

$$\sigma(x)\, q(x+\delta) > 0 \qquad \text{if} \quad x \in X_4, 0 < \delta \leqslant \delta_0. \tag{7}$$

Choose $\lambda > 0$ sufficiently small so that

$$q(x) \neq 0 \qquad \text{for all} \quad x \in \bigcup_{i=1}^{n-m+1} (w_i, w_i + \lambda].$$

Then, from (5), (6), (7), it follows that

$$\sigma(w_i)\, q(w_i + \lambda) > 0, \qquad i = 1, ..., n-m+1. \tag{8}$$

Let $(w_i, w_{i+1}] \cap S(f) = \{z_{i1}, ..., z_{iv_i}\}$, $i = 1, ..., n-m$. Next we shall show that q has at least $v_i + 1$ zeros in $(w_i, w_{i+1}]$ counting a double zero twice.

In fact, from the fact that $f - p$ alternates once in (w_i, w_{i+1}), it follows that $\sigma(w_{i+1}) = (-1)^{v_i+1}\, \sigma(w_i)$.

Suppose that q vanishes in $(w_i, w_{i+1}]$ only at $z_{i1}, ..., z_{iv_i}$ and each of these points is a simple zero. Then

$$\operatorname{sgn} q(w_{i+1} + \lambda) = (-1)^{v_i} \operatorname{sgn} q(w_i + \lambda).$$

From (8), it follows that $\sigma(w_{i+1}) = (-1)^{v_i}\, \sigma(w_i)$. This is a contradiction.

Thus q has at least $v_i + 1$ zeros in $(w_i, w_{i+1}]$ counting a double zero twice.

Now assume that there are v elements in $S(f) \backslash (w_1, w_{n-m+1}]$. Then $v + \sum_{i=1}^{n-m} v_i =$ the number of elements in $S(f) = m$. Hence q has at least $v + \sum_{i=1}^{n-m} (v_i + 1) = n$ zeros in $[a, b]$ and so $q \equiv 0$. This is a contradiction. Therefore p is a best copositive approximation.

($\Rightarrow$) Suppose that p is a best copositive approximation to f. By Theorem 3.10, there exist $n-m+1$ constants $\lambda_v > 0$, $v = 1, ..., n-m+1$, and $n-m+1$ distinct points $\{x_v\}_{v=1}^{N_0} \subset X_{+1} \cup X_{-1}$, $\{x_v\}_{v=N_0+1}^{N_1} \subset X_{+2} \cup X_{-2}$, $\{x_v\}_{v=N_1+1}^{N_2} \subset X_3$, and $\{x_v\}_{v=N_2+1}^{n-m+1} \subset X_4$, where $N_0 \geq 1$, such that

$$\sum_{v=1}^{N_1} \lambda_v \sigma(x_v)\, q(x_v) + \sum_{v=N_1+1}^{N_2} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v + \delta)}{\Phi(x_v + \delta)} + \sum_{v=N_2+1}^{n-m+1} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v - \delta)}{\Phi(x_v - \delta)} = 0$$

for all $q \in M_0$.

Let $W = \{x_1, ..., x_{n-m+1}\} = \{w_1, ..., w_{n-m+1}\}$, where $w_1 < \cdots < w_{n-m+1}$. Then $f - p$ alternates once in each (w_i, w_{i+1}), $i = 1, ..., n-m+1$.

In fact, if $f - p$ did not alternate once in some (w_{i_0}, w_{i_0+1}), $1 \leq i_0 \leq n-m$, let

$$Z_0 = \{s_i \in (w_i, w_{i+1}) \backslash S(f) : f - p \text{ alternates once in } (w_i, w_{i+1}),\ 1 \leq i \leq n-m\}.$$

If $Z = Z_0 \cup S(f)$, then Z contains less than $(n-m)+m=n$ elements.

For the remainder of our proof, using the same argument as the necessary part of Theorem 3.10, we will reach a contradiction. ∎

4. Uniqueness of Best Copositive Approximations

LEMMA 4.1. *If $q \in M_0 \backslash \{0\}$ has $n-1$ simple zeros in (a, b), then*

$$\lim_{\delta \to 0^+} \frac{q(z+\delta)}{\Phi(z+\delta)} \neq 0 \qquad \textit{and} \qquad \lim_{\delta \to 0^+} \frac{q(z-\delta)}{\Phi(z-\delta)} \neq 0$$

for all $z \in S(f)$.

Proof. Suppose

$$\lim_{\delta \to 0^+} \frac{q(z_0+\delta)}{\Phi(z_0+\delta)} = 0 \qquad \text{for some} \quad z_0 \in S(f).$$

Since $q \in M_0$ has $n-1$ simple zeros in (a, b), there are $x_1 < \cdots < x_n$ in (a, b) such that

$$q(x_i) \cdot q(x_{i+1}) < 0 \qquad \text{for} \quad i = 1, ..., n.$$

Assume that $z_0 \in (x_{k-1}, x_k)$ and, without loss of generality, assume that both q and Φ are positive in the interval $(z_0, z_0 + \lambda)$ for some suitable $\lambda > 0$.

Choose $\alpha > 0$ small enough such that

$$\alpha\, |\Phi(x_i)| < |q(x_i)| \qquad \text{for} \quad i = 1, ..., n.$$

We have

$$\lim_{\delta \to 0^+} \frac{(q - \alpha\Phi)(z_0+\delta)}{\Phi(z_0+\delta)} = -\alpha < 0,$$

so $(q-\alpha\Phi)(y) < 0$ for some $y \in (z_0, \min\{(z_0+\lambda), x_k\}$. But then $q - \alpha\Phi$ has a weak alternation of length $n+2$ in $x_1, ..., x_{k-1}, z_0, y, x_k, ..., x_n$, a contradiction. ∎

LEMMA 4.2. *Let $\lambda_v > 0$, $v = 1, ..., n-m+1$ be $n-m+1$ distinct constants, let $\{x_v\}_{v=1}^{N_1} \subset X_p^*(N_1 \geqslant 1)$, $\{x_v\}_{v=N_1+1}^{N_2} \subset X_3$, and $\{x_v\}_{v=N_2+1}^{n-m+1} \subset X_4$ be $n-m+1$ distinct points satisfying*

$$\sum_{v=1}^{N_1} \lambda_v \sigma(x_v)\, q(x_v) + \sum_{v=N_1+1}^{N_2} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v+\delta)}{\Phi(x_v+\delta)} + \sum_{v=N_2+1}^{n-m+1} \lambda_v \lim_{\delta \to 0^+} \frac{q(x_v-\delta)}{\Phi(x_v-\delta)} = 0$$

for all $q \in M_0$.

If $q \in M_0$ *satisfies*

$$q(x_v) = 0, \qquad v = 1, ..., N_1, \tag{9}$$

$$\lim_{\delta \to 0^+} \frac{q(x_v + \delta)}{\Phi(x_v + \delta)} = 0, \qquad v = N_1 + 1, ..., N_2, \tag{10}$$

$$\lim_{\delta \to 0^+} \frac{q(x_v - \delta)}{\Phi(x_v - \delta)} = 0, \qquad v = N_2 + 1, ..., n - m + 1. \tag{11}$$

Then $q \equiv 0$.

Proof. (1) If $N_1 = n - m + 1$, then $q(x_v) = 0$, $v = 1, ..., n - m + 1$. Since $x_v \in X_p^*$, $v = 1, ..., n - m + 1$, $x_v \notin S(f) = \{z_1, ..., z_m\}$, but $q(z_i) = 0$, $i = 1, ..., m$. So q has $n + 1$ zeros and thus $q \equiv 0$.

(2) If $N_1 < n - m + 1$, assume that $q \in M_0$ satisfies (9), (10), and (11), but $q \not\equiv 0$. Let

$$DZ(q) \cap \{x_{N_1+1}, ..., x_{N_2}\} = \{d_1, ..., d_{l_1}\},$$

$$DZ(q) \cap \{x_{N_2+1}, ..., x_{n-m+1}\} = \{d_{l_1+1}, ..., d_{l_2}\},$$

$$DZ(q) \setminus \{d_1, ..., d_{l_1}, d_{l_1+1}, ..., d_{l_2}\} = \{d_{l_2+1}, ..., d_{l_3}\},$$

$$SZ(q) = \{s_1, ..., s_k\}.$$

Then $k + 2l_3 \leqslant n - 1$.

Choose $\delta_0 > 0$ such that

$$q(x) \neq 0 \qquad \text{for} \quad x \in \bigcup_{i=1}^{l_3} [(d_i - \delta_0, d_i + \delta_0) \setminus \{d_i\})] \cup \bigcup_{i=1}^{k} [(s_i - \delta_0, s_i + \delta_0) \setminus \{s_i\})].$$

If

$$u = \min\{x_{N_1+1}, ..., x_{N_2}, x_{N_2+1}, ..., x_{n-m+1}\},$$

then $u \in (a, b)$.

Choose $l = n - 1 - k - 2l_3$ distinct elements $t_1, ..., t_l$ in

$$(a, u) \bigg\backslash \bigcup_{i=1}^{l_3} [(d_i - \delta_0, d_i + \delta_0)] \cup \bigcup_{i=1}^{k} [(s_i - \delta_0, s_i + \delta_0)].$$

If

$$T = \{s_1, ..., s_k, t_1, ..., t_l, d_1, ..., d_{l_3}, d_1 - \delta_0, ..., d_{l_1} - \delta_0, d_{l_1+1} + \delta_0, ..., d_{l_2} + \delta_0, d_{l_2+1} + \delta_0, ..., d_{l_3} + \delta_0\},$$

then T has $k + l + 2l_3 = n - 1$ elements.

Choose $q_0 \in M$ such that $q_0(x)=0$ for all $x \in T$. Let $q_1 = \sigma q_0$ and select $\sigma = +1$ or -1 appropriately so that

$$\operatorname{sgn} q(x_v + \delta_0) = \operatorname{sgn} q_1(x_v + \delta_0), \qquad v = N_1 + 1, ..., N_2,$$
$$\operatorname{sgn} q(x_v - \delta_0) = \operatorname{sgn} q_1(x_v - \delta_0), \qquad v = N_2 + 1, ..., n-m+1.$$

Choose $\lambda_0 > 0$ such that

$$\lambda_0 \, |q_1(x_v + \delta_0)| < |q(x_v + \delta_0)|, \qquad v = N_1 + 1, ..., N_2, \tag{12}$$
$$\lambda_0 \, |q_1(x_v - \delta_0)| < |q(x_v - \delta_0)|, \qquad v = N_2 + 1, ..., n-m+1. \tag{13}$$

If $q_2 = q - \lambda_0 q_1$, then $q_2 \not\equiv 0$ and $Z(q) \subset Z(q_2)$. So $q_2 \in M_0$.
Since q_1 has $n-1$ zeros, by Lemma 4.1,

$$\lim_{\delta \to 0^+} \frac{q_1(z \pm \delta)}{\Phi(z \pm \delta)} \neq 0 \qquad \text{for all} \quad z \in S(f).$$

Hence,

$$\lim_{\delta \to 0^+} \frac{q(x_v + \delta)}{q_1(x_v + \delta)} = 0, \qquad v = N_1 + 1, ..., N_2$$
$$\lim_{\delta \to 0^+} \frac{q(x_v - \delta)}{q_1(x_v - \delta)} = 0, \qquad v = N_2 + 1, ..., n-m+1.$$

Thus,

$$\lim_{\delta \to 0^+} \frac{q_2(x_v + \delta)}{q_1(x_v + \delta)} = -\lambda_0, \qquad v = N_1 + 1, ..., N_2$$
$$\lim_{\delta \to 0^+} \frac{q_2(x_v - \delta)}{q_1(x_v - \delta)} = -\lambda_0, \qquad v = N_2 + 1, ..., n-m+1.$$

Therefore there exists $0 < \delta_1 < \delta_0$ such that

$$\operatorname{sgn} q_2(x_v + \delta_1) = -\operatorname{sgn} q_1(x_v + \delta_1) = -\operatorname{sgn} q(x_v + \delta_0),$$
$$v = N_1 + 1, ..., N_2$$
$$\operatorname{sgn} q_2(x_v - \delta_1) = -\operatorname{sgn} q_1(x_v - \delta_1) = -\operatorname{sgn} q(x_v - \delta_0),$$
$$v = N_2 + 1, ..., n-m+1.$$

But from (12) and (13), we have

$$\operatorname{sgn} q_2(x_v + \delta_0) = \operatorname{sgn} q(x_v + \delta_0), \qquad v = N_1 + 1, ..., N_2$$
$$\operatorname{sgn} q_2(x_v - \delta_0) = \operatorname{sgn} q(x_v - \delta_0), \qquad v = N_2 + 1, ..., n-m+1.$$

So q_2 has at least one zero in each $(x_v+\delta_1, x_v+\delta_0)$, $v=N_1+1, ..., N_2$, and each $(x_v-\delta_0, x_v-\delta_1)$, $v=N_2+1, ..., n-m+1$. But $q_2\in M_0$ and

$$q_2(x_v)=0, \qquad v=1, ..., n-m+1.$$

Thus q_2 has $N_1+(n-m+1-N_1)+m=n+1$ zeros and $q_2\equiv 0$. This is a contradiction. ∎

THEOREM 4.3 (Strong Uniqueness). *Let $p\in M_f$ be a best copositive approximation to $f\in C[a, b]$. Then there is a positive constant $r=r(f)$ such that*

$$\|f-q\|\geqslant\|f-p\|+r\,\|p-q\|$$

for all $q\in M_f$. In particular, best copositive approximations are unique.

Proof. Let $p\in M_f$ be a best copositive approximation to f. By Theorem 3.10, there exist $\lambda_v>0$, $v=1, ..., n-m+1$, and $\{x_v\}_{v=1}^{N_0}\subset X_{+1}\cup X_{-1}$, $\{x_v\}_{v=N_0+1}^{N_1}\subset X_{+2}\cup X_{-2}$, $\{x_v\}_{v=N_1+1}^{N_2}\subset X_3$, $\{x_v\}_{v=N_2+1}^{n-m+1}\subset X_4$, where $N_0\geqslant 1$, such that

$$\sum_{v=1}^{N_0}\lambda_v\sigma(x_v)\,q(x_v)+\sum_{v=N_0+1}^{N_1}\lambda_v\sigma(x_v)\,q(x_v)$$
$$+\sum_{v=N_1+1}^{N_2}\lambda_v\lim_{\delta\to 0^+}\frac{q(x_v+\delta)}{\Phi(x_v+\delta)}+\sum_{v=N_2+1}^{n-m+1}\lambda_v\lim_{\delta\to 0^+}\frac{q(x_v-\delta)}{\Phi(x_v-\delta)}=0 \tag{14}$$

for all $q\in M_0$.

Let

$$H=\Bigg\{q\in M_0 : \sigma(x_v)\,q(x_v)\leqslant 0,\ v=N_0+1, ..., N_1;$$
$$\lim_{\delta\to 0^+}\frac{q(x_v+\delta)}{\Phi(x_v+\delta)}\leqslant 0,\ v=N_1+1, ..., N_2;$$
$$\lim_{\delta\to 0^+}\frac{q(x_v-\delta)}{\Phi(x_v-\delta)}\leqslant 0,\ v=N_2+1, ..., n-m+1\Bigg\}.$$

Obviously, for all $q\in M_f$, $-q\in H$, so $H\backslash\{0\}\neq\varnothing$.

Suppose there is a $q\in H\backslash\{0\}$ with $\sigma(x_v)\,q(x_v)\leqslant 0$, $v=1, ..., N_0$.

By (14), we have

$$q(x_v)=0, \qquad v=1, ..., N_1,$$
$$\lim_{\delta\to 0^+}\frac{q(x_v+\delta)}{\Phi(x_v+\delta)}=0, \qquad v=N_1+1, ..., N_2,$$
$$\lim_{\delta\to 0^+}\frac{q(x_v-\delta)}{\Phi(x_v-\delta)}=0, \qquad v=N_2+1, ..., n-m+1.$$

By Lemma 4.2, $q \equiv 0$. This contradicts the fact that $q \not\equiv 0$. Hence, for all $q \in H \backslash \{0\}$, there exists $w \in \{1, ..., N_0\}$ such that $\sigma(x_v)\, q(x_w) > 0$.

Since $H_0 = \{q \in H : \|q\| = 1\}$ is compact,

$$r = \inf\{\max_{1 \leqslant v \leqslant N_0} \sigma(x_v)\, q(x_v) : \|q\| = 1, q \in H\} > 0.$$

If $q \in M_f$ and $q \not\equiv p$, then $p - q \in H$. Therefore, there exists $v_0 \in \{1, ..., N_0\}$, such that

$$\sigma(x_{v0}) \frac{p(x_{v0}) - q(x_{v0})}{\|p - q\|} \geqslant r.$$

It follows that

$$\begin{aligned} r \, \|p - q\| &\leqslant \sigma(x_{v0})[p(x_{v0}) - q(x_{v0})] \\ &= \sigma(x_{v0})[f(x_{v0}) - q(x_{v0})] - \sigma(x_{v0})[f(x_{v0}) - p(x_{v0})] \\ &\leqslant \|f - q\| - \|f - p\|. \end{aligned}$$

Therefore,

$$\|f - q\| \geqslant \|f - p\| + r\, \|p - q\|$$

for all $q \in M_f$. ▮

For every $f \in C[a, b]$, let $\tau(f) \in M_f$ be the best copositive approximation to f. Unlike in the classical theory, this best copositive approximation operator is no longer continuous. An example showing this fact can be found in [9].

However, if we define

$$W_{f_0} = \{f \in C[a, b] : r(f) \in M_{f_0}\},$$

then a similar argument to the one in the classical case [1, p. 82] implies the following continuity theorem.

THEOREM 4.4. *To each $f_0 \in C[a, b]$ there corresponds a number $\lambda > 0$ such that for all $f \in W_{f_0}$,*

$$\|\tau f_0 - \tau f\| \leqslant \lambda \, \|f_0 - f\|.$$

5. Computation of Best Copositive Approximations

In this section, we suggest a method for computing the best copositive approximation. This method is based on converting copositive approximation to restricted range approximation, and then applying the algorithm developed for restricted range approximation to copositive approximation. A direct algorithm remains to be developed.

For $f \in C[a, b] \backslash M$, let $g_1, ..., g_k \in M_f$ be k linearly independent elements so that

$$\operatorname{span} M_f = \operatorname{span}\{g_1, ..., g_k\}.$$

Throughout this section, we shall assume that f does not vanish on any subinterval of $[a, b]$.

If p is the best copositive approximation to f, then $\|f - p\| \leqslant \|f\|$ and thus $\|p\| \leqslant 2\|f\|$.

Let

$$D = \left\{(c_1, ..., c_k) \in R^k \mid q(x) = \sum_{i=1}^{k} c_i g_i(x), \|q\| \leqslant 2\|f\|\right\},$$

$$C = \max_{(c_1, ..., c_k) \in D} \max_{1 \leqslant i \leqslant k} |c_i| \quad \text{and} \quad G(x) = \sum_{i=1}^{k} g_i(x).$$

Define

$$u_0(x) = \begin{cases} CG(x) & \text{if } x \in U(f) \\ 0 & \text{if } x \in L(f) \end{cases}$$

and

$$l_0(x) = \begin{cases} 0 & \text{if } x \in U(f) \\ CG(x) & \text{if } x \in L(f). \end{cases}$$

It is easy to check that u_0, $l_0 \in C[a, b]$. Let $u(x) = \max(u_0(x), f(x))$, $l(x) = \min(l_0(x), f(x))$ and $K_f = \{q \in M \mid l(x) \leqslant q(x) \leqslant u(x), x \in [a, b]\}$. It is clear that u, $l \in C[a, b]$, $l(x) \leqslant f(x) \leqslant u(x)$ and $K_f \subset M_f$.

THEOREM 5.1. *Let $f \in C[a, b]$. If f does not vanish on any subinterval of $[a, b]$, then the best copositive approximation to f is the same as the best approximation to f from $K_f = \{q \in M \mid l(x) \leqslant q(x) \leqslant u(x), x \in [a, b]\}$, where $u(x)$ and $l(x)$ are defined as above.*

Proof. The proof is fairly easy. It is left to the reader. ∎

Let $K_m = \{q \in M \mid l(x) \leqslant q(x) \leqslant u(x) + 1/m\}$, where m is an integer. Define

$$e_m = \|f - p_m\| = \inf_{q \in K_m} \|f - q\|$$

and

$$e = \|f - p\| = \inf_{q \in K_f} \|f - q\|.$$

Then we have the following theorem.

THEOREM 5.2. (a) *$p_m(x)$ converges uniformly to $p(x)$ as $m \to \infty$.*
(b) *$e_m \uparrow e$.*

Proof. Since $p \in K_f \subset K_m$, we have that

$$e_m = \|f - p_m\| \leqslant \|f - p\| = e \qquad \text{for all } m.$$

And since $M_{m+1} \subset M_m$,

$$e_m = \|f - p_m\| \leqslant \|f - p_{m+1}\| = e_{m+1}.$$

Thus $\{e_m\}_{m=1}^{\infty}$ is a monotone increasing sequence bounded by e.

Let $\{p_n\}$ be any subsequence of $\{p_m\}$. Since $l(x) \leqslant p_n(x) \leqslant u(x) + 1/n$, $\{p_n\}$ is uniformly bounded. Therefore there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ so that it converges uniformly to an element $p^* \in K_f$.

Let $e^* = \|f - p^*\|$. We shall show that $e^* = e$.

In fact, it follows from $p^* \in K_f$ that

$$e = \|f - p\| \leqslant \|f - p^*\| = e^*. \tag{15}$$

On the other hand, since p_{n_k} converges uniformly to p^*, $\|f - p_{n_k}\| \to \|f - p^*\|$. Thus

$$e^* = \|f - p^*\| = \lim_{k \to \infty} \|f - p_{n_k}\| = \lim_{k \to \infty} e_{n_k} \leqslant e. \tag{16}$$

Combining (15) and (16), we have $e = e^*$.

By the uniqueness of best copositive approximation, we have $p = p^*$. This means that any subsequence of $\{p_m\}$ contains a subsequence which converges uniformly to p. Hence p_m converges uniformly to p. Consequently, we have $e_m \uparrow e$. ∎

A Remes' type algorithm for computing p_m and e_m has been developed by Taylor and Winter in [13]. Now since one may compute p_m and e_m, and since $p_m(x) \to p(x)$ uniformly and $e_m \uparrow e$, we have obtained an algorithm for computing p and e.

ACKNOWLEDGMENTS

The author thanks Professor Frank Deutsch for his helpful suggestions and Professor Bruno Brosowski for conversations concerning the alternation theorem. He also thanks the referees for their comments and a shorter proof of Lemma 4.1.

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